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The limit equilibrium of an anisotropic medium under the general plasticity condition☆

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Abstract

A theory of the limit equilibrium of an anisotropic medium under the general plasticity condition in the plane strain state is developed. The proposed yield criterion (the limit equilibrium condition) is obtained by combining the von Mises–Hill yield criterion of an ideally plastic anisotropic material and Prandtl's limit equilibrium condition for a medium under the general plasticity law. It is shown that the problem is statically determinate, i.e., if the boundary conditions are specified in stresses, the stress state in plastic region can only be obtained using equilibrium equations. It is established that the equations describing the stress state are hyperbolic and have two families of characteristic curves that intersect at variable angles. In deriving the equations describing the velocity field, the material is assumed to be rigid plastic, and the associated law of flow is applied. It is shown that the equations for the velocities are also hyperbolic, and their characteristic curves are identical with those of the equations for stresses. However, the directions of the principal values of the stress and strain rate tensors are different due to the anisotropy of the material. The characteristic directions differ from the isotropic case in that the normal and tangential components of the stress tensor do not satisfy the limit conditions. It is established that the equations obtained allow of partial solutions, and in this case, at least one family of characteristic curves consists of straight lines. The conditions along the lines of discontinuity of the velocity are investigated, and it is shown that, as in the isotropic case, these are characteristic curves of the system of governing equations. In the anisotropic formulation, the well-known Rankine problem of the limit state of a ponderable layer is solved. From an analysis of the velocity field it is shown that plastic flow of the entire layer is possible only for a slope angle equal to the angle of internal friction. For slope angles less than the angle of internal friction, the solutions obtained are solutions of problems of the pressure of the medium on the retaining walls. The change in this pressure as a function of the parameters of anisotropy is investigated, and turns out to be significant.

According to established terminology, following Prandtl,[1] the mechanics of media under the general plasticity condition is understood as the science of the laws of deformation, under the action of external loads, of soils, rock, and granular and granulated media and other materials, the behaviour of which is characterized by the fact that the conditions for a transition to a state of plastic flow (the yield criterion) depend on the hydrostatic pressure and are termed the conditions of plasticity of general form. Physically this means that the medium changes to a plastic state if, at the point considered, an area exists on which the normal σ_n^* and tangential τ_n^* components of the stress tensor satisfy the limit condition $\tau_n^* = f(\sigma_n^*)$. In the isotropic mechanics of granular and plastic media, such areas are termed areas of slip, which, in the plane strain state, coincide in direction with the characteristic curves of the system of governing

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equations. The Coulomb criterion, widely used in the mechanics of granular media,[2] is a special case where the limit relation is a linear function.

Starting with the basic studies by Coulomb and up to the beginning of the 1960s, the mechanics of isotropic media under the general plasticity condition was developed chiefly as statics. This is due largely to the fact that the equations describing the stress state of a medium in the plane strain state,[2] like equations of the theory of ideal plasticity,[3] are statically determinate, i.e. in the case of statically determinate boundary conditions they can be solved without using kinematic relations. Here, the equations describing the stress state are hyperbolic and have two families of characteristic curves that are slip lines and intersect at angles which depend on the angle of internal friction of the medium.

Attempts to investigate the strain state of such media have generally been based on assumptions regarding the rigidly plastic behaviour of material and its incompressibility. The last assumption has led to equations for determining the velocity fields which also have two families of characteristic curves that prove to be mutually orthogonal and consequently do not coincide with the characteristic curves of the equations describing the stresses.

This contradiction was resolved within the framework of a flow theory[4] based on the application of the associated law of flow to the yield criterion. The principal advantage of this theory[4] is that the characteristic curves of the equations describing the stress and velocity fields in the plane strain state coincide, and the regions of limit equilibrium can be determined uniquely. On the basis of this model, a detailed investigation of the resolving system of equations and the lines of discontinuity of the velocity and stress was carried out,[5–8] a number of new problems were solved, including problems with mixed boundary conditions, and effective numerical methods were developed for solving the boundary-value problems.

The historic development has been such that the theories set out above, following Coulomb, are not entirely accurately called theories of limit equilibrium. Their principal task is to find the limit loads, interpreted as the loads at which a loss of equilibrium (a loss of stability) of the medium occurs. Essentially, from the modern viewpoint, they are theories of the flow of rigidly plastic material, and the limit loads are understood to be the loads at which such flow becomes possible. It is in this sense that the given terms are used below.

In well-known papers by Sedov, Il'yushin, Truesdell and Noll, the basic questions of non-linear continuum mechanics, associated with problems of symmetry and possible forms of writing of scalar, vector, and tensor functions invariant to certain orthogonal transformations, have undergone considerable development. An up-to-date exposition of these problems can be found in,[9] where extensions of flow theory and the deformation theory of plasticity to the case of finite strains and variation in temperatures are also given.

In determining the stress–strain state of a granular medium, as constitutive relations between the stress and strain tensor components, use is often made of relations similar to those of the theory of elasticity and the deformation theory of plasticity. As an example we can cite the classical papers,[10,11] in the first of which an elastoplastic model of soft soil is developed, subsequently extended to solid rock subject to intensive loads. Such approaches are widely used, especially in engineering applications,[12,13] to solve specific problems. Substantiation of the possibility of such an approach and its relation to the theory of limit equilibrium is examined in Ref. 14.

In spite of the extensive literature devoted to the mechanics of media under the general plasticity condition, problems related to taking the anisotropic properties of a medium into account regardless of elasticity have been inadequately investigated. A generally recognized theory of the deformation of an anisotropic medium that is similar to the von Mises–Hill theory for an anisotropic ideally plastic material has not so far been developed. This is not only of theoretical interest but also of practical importance, since most soils and rock possess pronounced anisotropy of mechanical properties.

1. Governing equations

In the anisotropic theory of ideal plasticity in the plane stress state, the stress tensor components satisfy the von Mises–Hill yield criterion[15]

$$T^2 = \tau_{xy}^2 + \frac{(\sigma_x - \sigma_y)^2}{4\lambda^2} = \tau_s^2; \quad 0 < \lambda < \infty \tag{1.1}$$

Here, the material is assumed to be rigidly plastic and orthotropic, the x and y axes coincide with the principal axes of anisotropy, τ_s is the yield point for pure shear in this system of coordinates and λ is the anisotropy parameter.

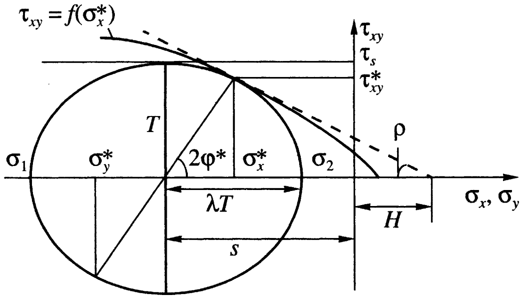

Fig. 1.

The limit condition (1.1) is satisfied identically if σ_x, σ_y and τ_{xy} are represented in the form

$$\sigma_{x,y} = s \pm \lambda\tau_s\cos 2\mu, \quad \tau_{xy} = \tau_s\sin 2\mu; \quad s = (\sigma_x + \sigma_y)/2 \tag{1.2}$$

If a stress plane is introduced such that the stress tensor components σ_x and σ_y are plotted along the abscissa axis and the component τ_{xy} is plotted along the ordinate axis, then, in such a system of coordinates, relations (1.2) represent the parametric equation of an ellipse tangential to the line $\tau_{xy} = \tau_s$ (Fig. 1). Note that the angle μ, defining the position of components (1.2) on the ellipse, is connected with the angle φ between the direction of the principal stress σ_1 and the x axis by the relation

$$\operatorname{tg} 2\varphi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{1}{\lambda}\operatorname{tg} 2\mu \tag{1.3}$$

If the material is isotropic ($\lambda = 1$), relation (1.1) becomes the von Mises yield criterion, the angles μ and φ will be equal, and the ellipse becomes a circle, where the magnitudes of the angles μ^* and φ^*, corresponding to the area on which the tangential stress τ_{xy}^* reaches its maximum value $\tau_{xy}^* = \tau_s$, are equal to $\mu^* = \varphi^* = \pi/4$.

We will assume that the anisotropic granular medium transfers to the limit state if at a point of the medium an area exists along which the normal component σ_x^* and the tangential component τ_{xy}^* of the stress tensor are related by the limit condition

$$\tau_{xy}^* = f(\sigma_x^*) \tag{1.4}$$

where $f(\sigma_x)$ is a known function, determined for each material experimentally.

In the general case, for an anisotropic medium under the general plasticity condition, we will assume that the magnitude of T in Eq. (1.1) is not constant but depends on the hydrostatic pressure

$$T = T(s) \tag{1.5}$$

and is defined such that the limit curve (1.4) is the envelope of the single-parameter family of ellipses (1.2). Here

$$\frac{dT}{ds} = -\frac{1}{\lambda}\sin\bar{\rho}, \quad \operatorname{tg}\bar{\rho} = \lambda \operatorname{tg}\rho \tag{1.6}$$

where ρ in the general case is the variable angle between the tangent to the limit curve (1.4) and the negative direction of the abscissa axis (Fig. 1). The form of the limit curve (1.4) and relation (1.6) enable us to obtain the required relation (1.5), after which formulae analogous to (1.2) will be written as

$$\sigma_{x,y} = s \pm \lambda T(s)\cos 2\mu, \quad \tau_{xy} = T(s)\sin 2\mu \tag{1.7}$$

and, after substituting them into the equilibrium equations, when there are no external forces, we obtain

$$\frac{\partial}{\partial x}(s + \lambda T(s)\cos 2\mu) + \frac{\partial}{\partial y}(T(s)\sin 2\mu) = 0, \quad \frac{\partial}{\partial x}(T(s)\sin 2\mu) + \frac{\partial}{\partial y}(s - \lambda T(s)\cos 2\mu) = 0 \tag{1.8}$$

If the boundary conditions of the problem are specified in stresses, then Eqs. (1.8), together with relation (1.5), enable us to determine completely the stress state and the corresponding limit loads. Consequently, the problem of the limit equilibrium of an orthotropic granular medium under the plasticity condition of general form is statically determinate.

Assuming that the function $T(s)$ [Eq. (1.5)] is the plastic potential, we obtain for the strain rate components

$$\dot{\varepsilon}_{x,y} = \frac{\nu}{2\lambda}(\sin\bar{\rho} \pm \cos 2\mu), \quad \dot{\gamma}_{xy} = \frac{\nu}{2}\sin 2\mu \tag{1.9}$$

The undefined factor ν is related to the magnitude of the maximum shear rate g as follows:

$$g = \frac{\dot{\varepsilon}_1 - \dot{\varepsilon}_2}{2} = \sqrt{\left(\frac{\dot{\varepsilon}_x - \dot{\varepsilon}_y}{2}\right)^2 + \dot{\gamma}_{xy}} = \frac{\nu\cos 2\mu}{2\lambda\cos 2\psi} \tag{1.10}$$

where 2ψ is the angle between the maximum strain rate ε_1 and the axis of anisotropy x, related to the angle φ [Eq. (1.3)] as follows:

$$\operatorname{tg} 2\psi = \frac{2\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \lambda^2 \operatorname{tg} 2\varphi \tag{1.11}$$

which indicates that, by virtue of the anisotropy of the medium, the principal axes of the stress and strain rate tensors do not coincide.

The rate of relative change in volume

$$e = \frac{\dot{\varepsilon}_x + \dot{\varepsilon}_y}{2} = \frac{\nu}{2\lambda}\sin\bar{\rho}$$

is related to the quantity (1.10):

$$e = g\sin\bar{\rho}\frac{\cos 2\psi}{\cos 2\mu} = -g\cos 2\alpha; \quad \cos 2\alpha = -\sin\bar{\rho}\frac{\cos 2\psi}{\cos 2\mu} \tag{1.12}$$

Since the angle $\alpha \geq \pi/4$, the shear strains cause an increase in volume – so-called dilatation.

Using formulae (1.9)–(1.11), the expressions for the components of the strain rate tensor in a t, n system of coordinates, rotated with respect to x, y by an angle β in the positive direction, can be written as

$$\dot{\varepsilon}_t, \dot{\varepsilon}_n = e \pm g\cos 2(\psi - \beta), \quad \dot{\gamma}_{tn} = g\sin 2(\psi - \beta) \tag{1.13}$$

By replacing the components of the strain rate tensor in formulae (1.9) by the components u and v of the velocity vector $\mathbf{v}$, we obtain an equation for the velocities in the form

$$2\lambda\sin 2\mu\frac{\partial u}{\partial x} = (\cos 2\mu + \sin\bar{\rho})\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \; 2\lambda\sin 2\mu\frac{\partial v}{\partial y} = (-\cos 2\mu + \sin\bar{\rho})\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{1.14}$$

It can be shown that, if $0 \leq \bar{\rho} \leq \pi/2$, Eqs. (1.8) and (1.14) are hyperbolic and have two coinciding families of characteristic curves. The expressions obtained are fairly cumbersome in form and are not given here.

2. A granular coulomb medium

In order to simplify subsequent calculations, we will assume that the limit condition (1.4) is the Coulomb criterion

$$\tau_{xy}^* = k - \sigma_x^* \operatorname{tg}\rho = (H - \sigma_x^*)\operatorname{tg}\rho \tag{2.1}$$

Note that, in the statics of granular media, the quantities k and $H = k \operatorname{ctg}\rho$ are called respectively the coefficient of adhesion and the reduced coefficient of adhesion, and ρ is the angle of internal friction. Relation (1.5) will take the form

$$T = \frac{1}{\lambda}(H - s)\sin\bar{\rho} \tag{2.2}$$

where $\operatorname{tg}\bar{\rho} = \lambda\operatorname{tg}\rho = \text{const}$.

The yield criterion, analogous to (1.1), will be

$$\lambda^2\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 = \left(H - \frac{\sigma_x + \sigma_y}{2}\right)^2 \sin^2\bar{\rho} \tag{2.3}$$

Equations similar to Eq. (1.7) can be written as follows:

$$\sigma_{x,\,y} = s \pm (H - s)\sin\bar{\rho}\cos 2\mu, \quad \tau_{xy} = \frac{1}{\lambda}(H - s)\sin\bar{\rho}\sin 2\mu \tag{2.4}$$

and Eq. (1.14), like formulae (1.9)–(1.13), will not change if we take into account the fact that in this case $\sin\,\bar{\rho} = \text{const}$.

Substituting expressions (2.4) into the equilibrium equations and introducing the replacement

$$(H - s) = H\,\mathrm{tg}\,\bar{\rho}\exp(2z\,\mathrm{tg}\,\bar{\rho})$$

we obtain

$$\begin{aligned}
&(1 - \sin\bar{\rho}\cos 2\mu)\frac{\partial z}{\partial x} - \frac{1}{\lambda}\sin\bar{\rho}\sin 2\mu\frac{\partial z}{\partial y} + \cos\bar{\rho}\sin 2\mu\frac{\partial \mu}{\partial x} - \frac{1}{\lambda}\cos\bar{\rho}\cos 2\mu\frac{\partial \mu}{\partial y} = 0\\
&\frac{1}{\lambda}\sin\bar{\rho}\sin 2\mu\frac{\partial z}{\partial x} - (1 + \sin\bar{\rho}\cos 2\mu)\frac{\partial z}{\partial y} + \frac{1}{\lambda}\cos\bar{\rho}\cos 2\mu\frac{\partial \mu}{\partial x} + \cos\bar{\rho}\sin 2\mu\frac{\partial \mu}{\partial y} = 0
\end{aligned} \tag{2.5}$$

Equations (1.14) and (2.5) are hyperbolic and have two families of characteristic curves that can be written as

$$\eta = z - M_+ = \text{const}, \quad \frac{dy}{dx} = -\frac{du}{dv} = \mathrm{tg}(\psi - \alpha) \tag{2.6}$$

$$\xi = z + M_- = \text{const}, \quad \frac{dy}{dx} = -\frac{du}{dv} = \mathrm{tg}(\psi + \alpha) \tag{2.7}$$

where

$$\begin{aligned}
&z = -\int\frac{ds}{2\,\mathrm{tg}\,\bar{\rho}(H - s)} = \frac{1}{2\,\mathrm{tg}\,\bar{\rho}}\ln\frac{H - s}{H\,\mathrm{tg}\,\bar{\rho}}\\
&M_{\pm} = \int\frac{\cos\bar{\rho}(\cos^2 2\mu + \lambda^2\sin^2 2\mu)d\mu}{\lambda\sqrt{\cos^2 2\mu + \lambda^2\sin^2 2\mu - \sin^2\bar{\rho}} \pm (\lambda^2 - 1)\sin\bar{\rho}\sin 2\mu\cos 2\mu}
\end{aligned} \tag{2.8}$$

Graphs of $M_{\pm}(\mu)$ are shown in Fig. 2 for a number of values of λ.

Note that in the isotropic case ($\lambda = 1$) $M_{\pm} = \mu = \phi$, and Eqs. (1.8) and (1.14) reduce to the analogous equations in Ref. 16. The characteristic curves (2.6), corresponding to the value of the characteristic parameter $\eta = \text{const}$, are normally called the characteristic curves of the first family. The characteristic curves (2.7) – $\xi = \text{const}$ – are called the characteristic curves of the second family. Similar to the analogous equations of the isotropic theory of ideal plasticity

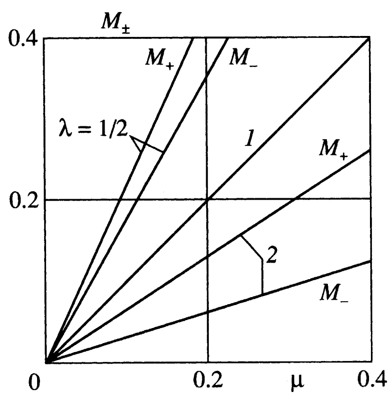

Fig. 2.

and the mechanics of a granular medium, Eqs. (2.6) and (2.7) allow of particular solutions corresponding to constant values of the characteristic parameters η and ξ.

If, in a certain region $\eta = \eta_0 = \text{const}$, and quantity ξ is variable, then along each of the characteristic curves of the second family (2.7) the quantities ψ, μ and α are constant and they will be straight lines. Here, Eq. (2.7) can be integrated, and their integrals are written as

$$\begin{aligned} &X = y\cos(\psi+\alpha) - x\sin(\psi+\alpha) = X(\xi), \quad V = u\cos(\psi+\alpha) + \upsilon\sin(\psi+\alpha) = V(\xi) \\ &z - M_+ = \eta_0, \quad M_+ + M_- = \xi + \eta_0 \end{aligned} \qquad (2.9)$$

where $X(\xi)$ and $V(\xi)$ are arbitrary functions that are determined by the corresponding boundary conditions. The characteristic curves of the first family in the x, y plane and the u, υ hodograph plane are curvilinear, and they intersect the straight lines (2.9) at angles 2α.

If, in a certain region, $\xi = \xi_0 = \text{const}$, and the quantity η is variable, the integrals of Eq. (2.6) will be

$$\begin{aligned} &Y = y\cos(\psi-\alpha) - x\sin(\psi-\alpha) = Y(\eta), \quad U = u\cos(\psi-\alpha) + \upsilon\sin(\psi-\alpha) = U(\eta) \\ &z + M_- = \xi_0, \quad M_+ + M_- = \xi_0 + \eta \end{aligned} \qquad (2.10)$$

i.e. the characteristic curves of the first family will be straight lines.

When, in the plastic region, both characteristic parameters $\eta = \eta_0$ and $\xi = \xi_0$ are constant, then both characteristic families are parallel straight lines, the lines of the first and second families intersect at a constant angle $2\alpha_0$, and the stress state in this region is constant. Note that the quantities U [Eq. (2.10)] and V [Eq. (2.9)] introduced are orthogonal projections of the velocity vector **v** onto the directions of the first and second families of characteristic curves respectively.

We will derive a condition on the lines along which the stresses are continuous and the velocity vector is discontinuous. Consider such a line with a tangential axis t and a normal axis n. We will assume that, on passing through the line of discontinuity, both the tangential component υ_t and, owing to dilatation (1.12), the normal component υ_n of the velocity, may be discontinuous. It is clear that the velocities υ_t and υ_n change very little along this line, and therefore it can be assumed that

$$\dot{\varepsilon}_n = \partial\upsilon_n/\partial n, \quad \dot{\varepsilon}_t = 0, \quad 2\dot{\gamma}_{tn} = \partial\upsilon_t/\partial n$$

From relations (1.13), if $\dot{\varepsilon}_t = 0$, we obtain

$$\operatorname{tg}\beta = \operatorname{tg}(\psi \pm \alpha), \quad \beta = \psi \pm \alpha \qquad (2.11)$$

i.e. the lines of discontinuity should coincide with one of the characteristic curves. Then the remaining strain rates are given by the following equations

$$\partial\upsilon_n/\partial n = -2g\cos 2(\psi - \beta), \quad \partial\upsilon_t/\partial n = \pm 2g\sin 2(\psi - \beta) \qquad (2.12)$$

From these relations we obtain

$$\delta\upsilon_n/\delta\upsilon_t = \mp\operatorname{ctg}2(\psi - \beta) = \mp\operatorname{ctg}2\alpha$$

where $\delta\upsilon_n$ and $\delta\upsilon_t$ are the normal and tangential components of the velocity jump $\delta\mathbf{v}$.

Let the lines of discontinuity of the velocity belong to the family of characteristic curves $\eta = \text{const}$ (Fig. 3). If the angle 2α is represented in the form $2\alpha = \pi/2 + \nu$, we can write

$$\delta\upsilon_n/\delta\upsilon_t = \operatorname{tg}\nu$$

i.e. the vector $\delta\mathbf{v}$ is inclined to the line of discontinuity at an angle ν and makes a right angle with the line of the family $\xi = \text{const}$. Consequently, the vector $\delta\mathbf{v}$ has a zero projection onto the line $\xi = \text{const}$, and the projection of the velocity vector V (2.9) onto this line changes continually.

If the line of discontinuity of the velocity belongs to the family $\xi = \text{const}$, we have

$$\delta\upsilon_n/\delta\upsilon_t = -\operatorname{tg}\nu$$

and the vector $\delta\mathbf{v}$ has a zero projection onto the line $\eta = \text{const}$, i.e. the projection of the velocity vector U [Eq. (2.10)] onto this line changes continually on passing through the line of discontinuity.

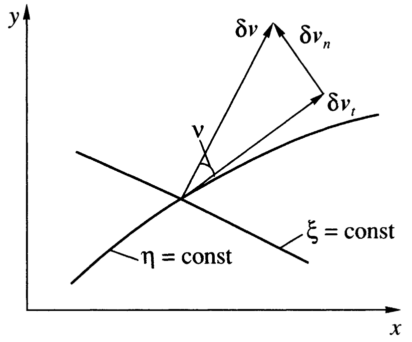

Fig. 3.

3. The limit equilibrium of a ponderable layer

Consider a layer of height h of orthotropic granular material lying on a rigid base, loaded by bulk forces of its own weight γ, the free surface of which material makes an angle θ with the horizontal and is stress-free. In the isotropic case, the problem of the limit equilibrium of such a layer was first formulated and solved by Rankine[17] and examined in detail by Nadai.[18] The solution of this problem can be obtained in simple and graphic form, if it is assumed that the material is ideally granular, i.e. in formulae (2.1)–(2.4) the quantities $k = H = 0$.

We will introduce a rectangular system of coordinate whose y axis is directed upwards along the layer, and whose x axis is directed into the interior of the layer (Fig. 4). Taking into account that the stress state is independent of the y coordinate and on the free surface $x = 0$ we have $\sigma_x = \tau_{xy} = 0$, from the equilibrium equations we find

$$\sigma_x = -\gamma x \cos\theta, \quad \tau_{xy} = \gamma x \sin\theta \tag{3.1}$$

Putting $H = 0$ in formulae (2.4), and equating the corresponding stress tensor components to the values in Eq. (3.1), to find the required functions s and μ we obtain the two equations

$$s\omega\sin 2\mu = -\lambda\gamma x\sin\theta, \quad s(1-\omega\cos 2\mu) = -\gamma x\cos\theta, \quad \omega = \sin\bar{\rho} \tag{3.2}$$

which have two sets of solutions

$$\sin 2\mu_{a,p} = \omega^{-1}\sin\chi(\cos\chi \pm \sqrt{\omega^2 - \sin^2\chi}), \quad s_{a,p} = -\gamma x\cos\theta/(1-\omega\cos 2\mu_{a,p}) \tag{3.3}$$

where we have introduced the notation $tg\,\chi = \lambda\, tg\,\theta$. It is not difficult to show that, if $\theta < \rho$, then $\omega^2 > \sin^2\chi$.

In the mechanics of granular media, the stress state corresponding to the plus sign in the first of Eq. (3.3) is called the minimum, or active, stress state, and that corresponding to the minus sign is called the maximum, or passive, stress state. The meaning of these terms will be clear from the solution of the problem for the velocities.

The first of formulae (3.3) indicates that the angle μ is independent of the x coordinate, and therefore each of the two families of characteristic curves consists of parallel lines, the lines of the first and second families intersect at an angle 2α. From this formula it also follows that, when $\chi = \bar{\rho}$ and $\theta = \rho$, i.e. when the slope is equal to the angle of

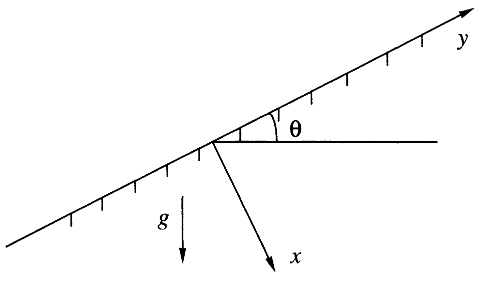

Fig. 4.

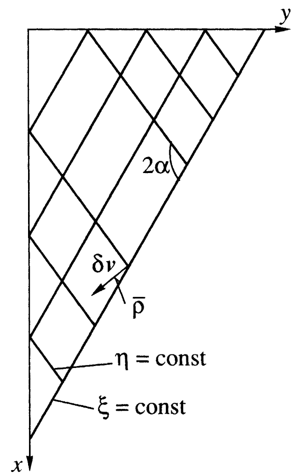

Fig. 5.

internal friction, the two sets of solutions are identical:

$$\mu_a = \mu_p = \mu^* = \pi/4 - \bar{\rho}/2, \quad s_a = s_p = -\gamma x \cos\rho/(1-\omega^2) \tag{3.4}$$

and (when $H=0$) the stress tensor components σ_x^* and τ_{xy}^* satisfy Coulomb's criterion (2.1). Here, the quantities ψ^* and α^*, which define the slopes of the characteristic curves, will be as follows:

$$\psi^* = \pi/4 - \rho/2, \quad \alpha^* = \pi/4 + \rho/2$$

Consequently, as in the isotropic case,[18] the characteristic curves of the first family (2.6) are vertical lines parallel to the gravity force, while the characteristic curves of the second family (2.7) are straight lines parallel to the slope.

Let us examine the second extreme case where the slope is zero. In this case,

$$\mu_a = \psi_a = \pi/2, \quad \alpha_a = \pi/4 + \bar{\rho}/2, \quad s_a = -\gamma x/(1+\omega), \quad \sigma_y^a = -\gamma x(1-\omega)/(1+\omega) \tag{3.5}$$

and the corresponding stress state will be active. The field of characteristic curves is shown in Fig. 5.

For the second set of solutions, from relations (3.3) we obtain

$$\mu_p = \psi_p = 0, \quad \alpha_p = \alpha_a, \quad s_p = -\gamma x/(1-\omega), \quad \sigma_y^p = -\gamma x(1+\omega)/(1-\omega) \tag{3.6}$$

and the stress state will be passive. The field of characteristic curves is shown in Fig. 6.

When the slope $\theta < \rho$, the lower boundary of the slope $x = h$ is not a characteristic curve, and consequently the velocities on it should be continuous, i.e. $u = v = 0$. Since these conditions define the Cauchy problem for homogeneous Eq. (1.14), in the entire layer the velocity of motion of the medium will be zero. Consequently,

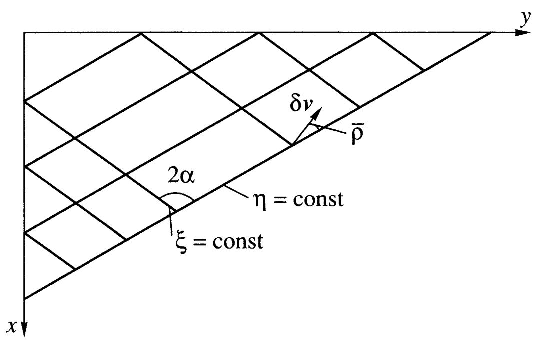

Fig. 6.

it cannot be assumed that the solutions (3.3), like (3.5) and (3.6), hold for the entire length of the slope. However, they can be interpreted as the solution of the problem of the pressure of the medium on the retaining wall.

For the case $\theta = 0$, the velocity field can be plotted if it is assumed that the plane $y = 0$ is the surface of an ideally smooth retaining wall. If the medium shifts the wall, and it moves to the left, then the stress state is described by formulae (3.5). The field of characteristic curves is shown in Fig. 5. Here, the characteristic curve of the family $\xi = \text{const}$, enclosing the plastic region and passing through the point $y = 0$, $x = h$ (the lower edge of the wall), will be the line of discontinuity of the velocity. Along this line the projection of the velocity vector U [equation (2.10)] is continuous and consequently equal to zero, while the total velocity vector $\delta\mathbf{v}$, with which the plastic region moves as a rigid whole, makes a constant angle $\bar{\rho}$ with the line of discontinuity.

However, if the wall shifts the medium and moves to the right, then the stress state in the medium is described by formulae (3.6), and the line of discontinuity of the velocity enclosing the plastic region belongs to the family of characteristic curves $\eta = \text{const}$. The field of characteristic curves and the direction of motion of the plastic region are shown in Fig. 6.

The force of the pressure of the medium on the wall for these cases is determined by the stresses σ_y^a [Eq. (3.5)] or σ_y^p [Eq. (3.6)] and accordingly is termed active (Na) or passive (Np). The numerical values of these forces for an angle of internal friction such that $\sin\rho = 1/3$ and for a number of values of λ are presented below:

λ	0.5	1	2
$\sin\bar{\rho}$	0.17	0.33	0.58
$2\text{Na}(\gamma h^2)$	0.7	0.5	0.27
$2\text{Np}(\gamma h^2)$	1.42	2	3.73

The value $\lambda = 1$ corresponds to the classic Rankine solution[18] for an isotropic granular medium.

Note that the motive force in an active stress state is the force of the weight of the slipping mass of granular material, the work of which will be expended on energy dissipation through the forces of internal friction along the line of discontinuity of the velocity and on the work of counteraction of the wall. In the case of a passive stress state, the motive force will be the force of the pressure of the retaining wall, the work of which will, as before, be expended in dissipating energy along the line of discontinuity of the velocity and changing the potential energy of the mass of granular material being shifted.

In a similar way, using formulae (3.3), the stress states can be obtained and the velocity fields in the vicinity of the retaining wall for the slope $0 < \theta < \rho$ can be plotted.

If the slope is equal to the angle of internal friction, then the stress state is given by formulae (3.4), and in this case the rigidly plastic boundary $x = h$ is a characteristic curve of the family $\xi = \text{const}$, which is parallel to the slope. This will be the line of discontinuity of the velocity, and the velocity vector of the motion of the medium makes angle ρ with it. The layer, over its entire length, moves at constant velocity in the horizontal direction. Consequently, the entire layer transfers into a state of plastic flow only when its slope is equal to the angle of internal friction; for smaller slopes, plastic flow is only possible in confined regions adjacent to the retaining wall. The velocity field analysis conducted and the proposed interpretation of the solution of Rankine's problem also hold for the isotropic case; this has not been described in the literature.

In conclusion, we note that, when well-known methods and approaches[19] are used, Eq. (1.8) can be integrated, and, using the theory of limit equilibrium, a more complete deformation type theory of elastoplastic deformation of anisotropic media under the general plasticity condition can be constructed.

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